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The Pedagogy-Andragogy-Heutagogy Spectrum in Secondary Mathematics Education: A Theoretical and Practical Review

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Abstract: As mathematics education evolves to meet the demands of Industry 4.0, there is a critical shift from teacher- led instruction toward self-determined learning models. This paper explores the application of the Pedagogy-Andragogy-Heutagogy (PAH) spectrum in secondary mathematics to foster dynamic, engaging learning outcomes. The study utilizes a theoretical review and a practical analysis of five instructional experiences, supported by 38 peer-reviewed sources published between 2020 and 2026. The findings indicate that transitioning along the PAH spectrum significantly enhances mathematical capability and learner autonomy. Practical evidence suggests that while pedagogical scaffolding provides a necessary foundation, moving toward heutagogical research papers increases student engagement and conceptual depth. The study concludes that a balanced PAH approach is vital for preparing secondary students for higher education and the modern workforce. Furthermore, institutional support for teacher professional development in these methodologies is essential for sustainable educational reform.

Keywords: Physical Literacy; Movement Education Program; Cognitive Engagement; Sustainable Development Goals; Cognitive Determinants; Quality Education; Effect Size.

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1. Introduction

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1.1. Status Quo on Mathematics Education

Mathematics education serves as a fundamental pillar for developing scientific reasoning and critical thinking skills from an early age [1]. Unlike many other subjects, mathematics is highly sequential: understanding one concept lays the foundation for mastering subsequent topics. This interconnected structure enables the application of mathematical models to a wide range of real-world problems [2]. Such competencies are particularly important for students pursuing Science, Technology, Engineering, and Mathematics (STEM) fields in higher education [3]. However, international assessments such as the Program for International Student Assessment (PISA) and the Trends in International Mathematics and Science Study (TIMSS) continue to reveal significant performance gaps in mathematics achievement, which are often associated with students' socioeconomic backgrounds [4]. Traditional teaching practices often assume that knowledge can be transmitted uniformly to all learners, with students expected to adapt to this mode of instruction [5]. However, some students perceive mathematics as an abstract subject, which necessitates the development of alternative teaching strategies [6]. Additionally, students' difficulties in mathematics cannot be attributed to a single cause, underscoring the need for educators worldwide to adopt a more holistic, adaptive approach to teaching that considers cognitive, pedagogical, and contextual factors influencing student learning [1].

Beyond the academic world, the acceleration of digitalization, artificial intelligence, and data- driven decision-making across all sectors of society demands not only mathematical literacy but also the ability to comprehend and process unfamiliar problems into numerical concepts [2]. This practical prospect underscores the need for critical re-examination of how mathematics is taught and experienced in secondary classrooms worldwide. Educators and policymakers are thus challenged to shift traditional learning models toward a more curiosity-driven approach to mitigate learners' tendency to be passive recipients of information [3]. This passive behavior may hinder students' ability to perform effectively in higher levels of education, not only in mathematics but across their broader academic development. Therefore, this review critically evaluates the practicality of shifting from teacher-centered (pedagogical) to student-centered, self-determined (heutagogical) learning to address and potentially alleviate this issue. This approach aims to foster students' curiosity, enable them to explore diverse mathematical concepts, and connect them with those taught in a classroom setting under the teacher's guidance, thereby enhancing creativity, problem-solving skills, and a sense of personal ownership in their learning [4].

1.2. The Evolving Role of the Learner

The role of learners in mathematics education has evolved over the last few decades. Students have developed a way to express their ideas by internalizing concepts with their own understanding, rather than passively receiving information from the teacher [7]. This shift reflects a dynamic process that needs to be encouraged and implemented in the education system to be constructive for both teachers and students. Educational research strongly highlights the significant correlation between learners' agency, metacognitive awareness, and self-regulation, and learners' performance in understanding and applying mathematical concepts. Hattie [14] synthesis of meta-analyses indicates that student agency and self-efficacy are among the most significant factors in achieving mathematical success. Given that learners themselves are the main drivers of learning, their self-perceptions should also help teachers adjust and develop their coursework. At the secondary level, redesigning the coursework would better align with adolescents' nature. Adolescent learners are at a developmental stage characterized by increasing autonomy, identity formation, and sensitivity to the relevance of academic content, during which the endless questions of "why" and "how" enhance their sense of involvement and positively affect their learning [4]. Instructional approaches that fail to acknowledge and capitalize on these developmental realities risk disengagement and surface-level processing. Conversely, approaches that honor learner agency and connect mathematical concepts to lived experience demonstrate measurable gains in motivation and deep conceptual understanding [8].

1.3. Defining the PAH Spectrum

Blaschke introduced the Pedagogy-Andragogy-Heutagogy (PAH) spectrum [4] and further developed it to align with digital and technological advancements, promoting lifelong learning. The PAH spectrum provides a comprehensive framework for understanding the instructional orientations involved in a typical learning process in which the learner shifts from a dominant role to multiple interactive roles of equal significance [9]. Pedagogy, derived from the Greek for 'leading children,' historically refers to teacher-directed instruction wherein the educator determines learning objectives, content, pace, and assessment. In short, pedagogy broadly encompasses any structured, teacher-facilitated learning process, regardless of learner age. Andragogy, coined by Knowles et al. [20] as the 'art and science of helping adults learn,' emphasizes the learner's prior experience, intrinsic motivation, and readiness to learn, as a stage in which ideas are internalized into a broader framework. The andragogical approach in mathematics involves collaborative inquiry, problem- solving in context-relevant scenarios, and learner participation in goal-setting [10]. The instructor transitions from instructor to facilitator, creating conditions for learning to branch out rather than be directed down a single pathway. Heutagogy, the most recent and radical point on the spectrum, was proposed by Black and William [3] and refers to self-determined learning. Learners are now independent and are expected to create their own pathways towards a learning goal they set for themselves in mathematical terms. Heutagogy manifests as student-initiated investigations, real-world problem framing, and deep reflective practice that integrates mathematical understanding with a broader spectrum of real-world applications [11]. The PAH spectrum thus represents a progression from dependence to independence, from content coverage to competence development, fostering learners' ongoing role in any field they enter.

1.4. The Problem in Secondary Mathematics

While student-centered approaches can undeniably shift students from a passive role, secondary mathematics classrooms in many institutions still lean towards pedagogical models. Teacher-based instruction, routine procedural practices, and summative evaluations such as tests and homework shift learning into an outcome to be achieved rather than something that builds deep conceptual understanding [12]. This contradicts what many education researchers have long advocated, suggesting that such practices may be rooted in deeply embedded historical traditions of teaching and learning. The consequences of this misalignment are observable across several dimensions. First, student motivation and engagement in mathematics decline significantly across the secondary years, a trend observed by Eccles and Wigfield [10]. Second, performance on tasks involving application- and problem-solving-related scenarios is markedly weaker than on routine procedural tasks. This harms the student’s ability to perform well in higher education and in the competitive professional world. Third, students from historically underrepresented groups still struggle to receive equality in authority-dependent classroom settings. These issues require not merely incremental instructional adjustments, but a fundamental reconfiguration of the roles of teachers and learners in mathematics education. The PAH spectrum provides such a framework for rethinking instructional design, while preserving flexibility in the distribution of roles and responsibilities between teachers and learners [13].

1.5. Theoretical Framework

This study is grounded in the synergistic combination of constructivist, humanistic, and complexity theories of learning [14]. Constructivist theory holds that learning is a process of actively thinking and building understanding, which is enhanced by social interaction and prior knowledge [17]. Humanistic theory (Maslow, Rogers) suggests that learning depends on the learner’s needs for safety and belonging, indicating the significance of nurturing emotional needs to foster better learning [15]. Lastly, complexity theory highlights that learning is not a “one-size fits all” approach; it is adaptive and messy. Together, these theories indicate that an effective mathematics pedagogy must be adaptive, responsive, and ultimately redistribute cognitive authority to the learner. Through these frameworks, the PAH spectrum is a sequential yet flexible strategy that allows teachers to apply their learning experiences to their students’ developmental readiness [16].

1.6. Research Aim

This paper aims to examine the theoretical and practical application of the PAH spectrum in secondary mathematics education. Specifically, the objectives are as follows:

- Establish a Theoretically Grounded Understanding of Pedagogy, Andragogy, And Heutagogy to Be Applied for Secondary Mathematics.
- Analyze Five Documented Instructional Experiences through the Lens of the PAH Spectrum.
- Evaluate The Implications of PAH-Informed Instruction for Student Engagement, Mathematical Understanding, and Learner Autonomy; and Offer Evidence-Based Recommendations for Practitioners and Institutional Leaders Committed to Transforming Secondary Mathematics Education.

2. Literature Review

2.1. Pedagogical Foundations in Mathematics: The Scaffolding Phase

In the PAH spectrum, pedagogy serves as the foundational layer upon which learning is first constructed [18]. This teacher-centered approach is still essential for students to develop procedural fluency, learn notation, and build concepts grounded in understanding for higher-order mathematical reasoning. Students who are given explicit instruction, worked examples, and structured practice for an entirely new topic perform better than those who lack this basic guidance [19]. The cognitive load theory suggests that a firm foundation in learning is often achieved through guidance and instruction, and only then are students equipped for self-learning [20]. This demonstrates that a teacher’s role in a pedagogical approach is undeniably important, especially in secondary mathematics, where formal proof, algebraic manipulation, and geometric reasoning depend on precise mathematical language and technique [21]. There are three distinct zones of Vygotsky's [33] proximal development (ZPD), each corresponding to the student’s level of learning independence [22]. There are three zones: independent (learning based on prior knowledge), not independent (new topics), and independent (with adequate support) [23]. The key challenge here is calibrating this support so that it builds capacity rather than creates dependency. Assessments within the pedagogical approach are not intended solely for evaluating students' performance [24]. They also enable teachers to identify areas where students excel and where instructional delivery may need improvement. At the same time, assessments provide students with opportunities to reinforce their understanding and improve through feedback [25]. Due to the negative stigma often associated with formal assessments, low-stakes classroom activities such as exit tickets, peer review, and immediate corrective feedback have been shown to promote greater mathematical progress [27]. Consequently, the pedagogical phase establishes the foundational conditions necessary for meaningful progression toward andragogy and heutagogy [26].

2.1.1. Practical Evidence: Experience 1 — Pythagorean Theorem

The first instructional experience illustrates a highly structured pedagogical approach to introduce the Pythagorean Theorem in a secondary mathematics class [27]. The lesson was guided by a detailed Lesson Note distributed at the beginning of the class, so that the students know what to expect and the objectives by the end of the class [28]. This approach aligns with what Kirschner et al. [19] term 'guided instruction,' wherein the instructional sequence is transparent, allowing students to orient their attention toward mathematical reasoning rather than procedural uncertainty (Figure 1).

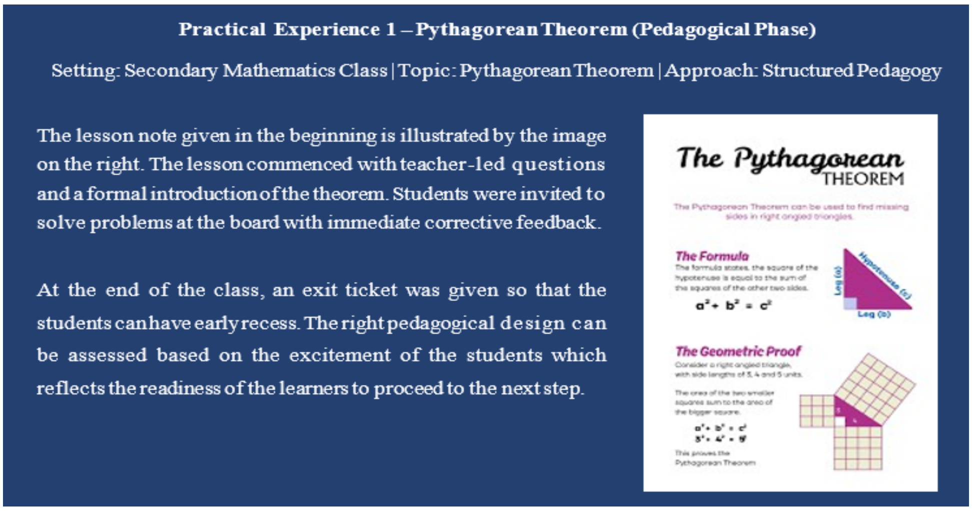

Figure 1: Lesson note example

By introducing a question in the beginning, it activates prior knowledge before introducing new content (Ausubel's advanced organizer model, as cited in Mayer [24]. The lesson progressed through the following sequence: defining the theorem, identifying the hypotenuse, and applying the theorem to find the unknown sides. Learning was directly applied through board work, where feedback was effectively given to address errors, aligning with the suggestions of Hattie and Timperley [15]. The exit ticket serves as a motivator for students to earn their recess through demonstrated competence.

2.1.2. Assessment in Pedagogy: Experience 1 — Probability (Written Assessment)

A practical example of pedagogical assessment is the written examination on a unit chapter aligned to the Cambridge 0850 curriculum [28]. These kinds of assessments are often linked to teacher-directed pedagogy. Students are expected to answer structured questions with point allocations depending on the level of critical thinking required to solve the problem. This design prioritizes measurement validity through point earnings and curriculum alignment [29]. The Cambridge curricula align closely with the pedagogical scheme in the PAH spectrum, highlighting the content standardization and external benchmarking required for secondary education [30]. The pedagogical end of the spectrum is demonstrated by its definition of what counts as mathematical knowledge, which is determined from the student's work as validation of correct understanding [32]. Despite limiting self-learning, this approach serves an important function by establishing shared standards to ensure students' eligibility for certification [31]. However, the assessment itself is not limited to reinforcing learning in students but also serves as a reflection on how teachers can adapt their teaching to a more andragogical approach, which is desirable for deeper, more lasting learning [33].

2.2. Moving Towards Andragogy: Relevance and Group Inquiry

Andragogy according to Knowles et al. [20] and as adapted for secondary learners by Taylor and Hamdy [31] are summarized by these assumptions: learners are motivated by the relevance and applicability of the content to their everyday lives; prior experiences are rich resources for further learning; learners prefer problem-centered over content-centered approaches; and that learners are capable of generating their own ideas in their own terms. These principles were originally applied to adult learners, but a more relevant candidate would be adolescents, particularly those engaged in STEM subjects. Given the interconnected nature of STEM concepts, perceived relevance is a critical factor for sustained engagement. In secondary mathematics, the andragogical approach is most visibly realized through collaborative problem-solving, project-based learning, and group discussions. These methods align with the andragogical concept of a multi-disciplinary approach to fulfill knowledge requirements in real-world situations. The teacher's role shifts from an "instructing mode" to a "facilitating mode," designing problems and collaborating with students in a supporting role rather than as a source. This way, mathematical rigor is maintained even as the learner's autonomy expands.

Despite this, the challenge lies in maintaining the mathematical depth and conceptual framework while simultaneously allowing students to engage and solve problems their way (Lampert [21], as cited in Smith and Stein [29]). Research on interdisciplinary learning underscores the particular power of andragogical approaches in creating conditions for what Boaler [5] calls 'mathematical connections,' the ability to see mathematical ideas as richly interconnected rather than as isolated procedures. When students encounter mathematics as a tool for investigating phenomena that matter to them — epidemiology, social statistics, economic decision-making — they develop both deeper conceptual understanding and a more robust mathematical identity. Research on interdisciplinary learning highlights the effectiveness of andragogical approaches in fostering connections in mathematics, enabling students to integrate and relate diverse mathematical concepts when solving complex problems. When mathematics is viewed and used as a tool for investigating real-world phenomena meaningful to learners, deeper conceptual understanding emerges, and mathematics shifts from being perceived as a discrete subject to becoming part of the learner's intellectual process.

2.2.1. Practical Evidence: Experience 2 — Interdisciplinary Project-Based Learning (IDPBL)

Students were given disease outbreak data and tasked as investigators to identify the root cause and prevent further transmission. In Mathematics, they constructed frequency tables and bar graphs of symptoms and illness counts over time. These mathematical outputs supported pathogen identification in Biology, while English complemented the investigation through report and creative writing tasks. Global perspectives compare how this disease affects different communities. This authentic purpose is a hallmark of andragogical design. The IDPBL experience represents one of the most sophisticated andragogical designs in the reported instructional repertoire. Eight subjects — English, Global Perspectives, Sciences, Mathematics, Bahasa Indonesia, Computer Science, and Visual Arts — converged around a shared investigative scenario involving a disease outbreak. This design reflects the distinguished features of well-designed project-based learning as described by Bonk and Zhang [6]: a driving question of genuine complexity (i.e. Why did the outbreak happen?), cross-disciplinary knowledge integration (multiple skills from various subjects), student agency within defined parameters (analyze problems and make conclusions), and authentic outputs with real communicative purpose (final presentation of the report). Within the mathematics component, students were tasked with constructing frequency tables and bar graphs from outbreak data as taught in the Probability and Statistics unit (Figure 2).

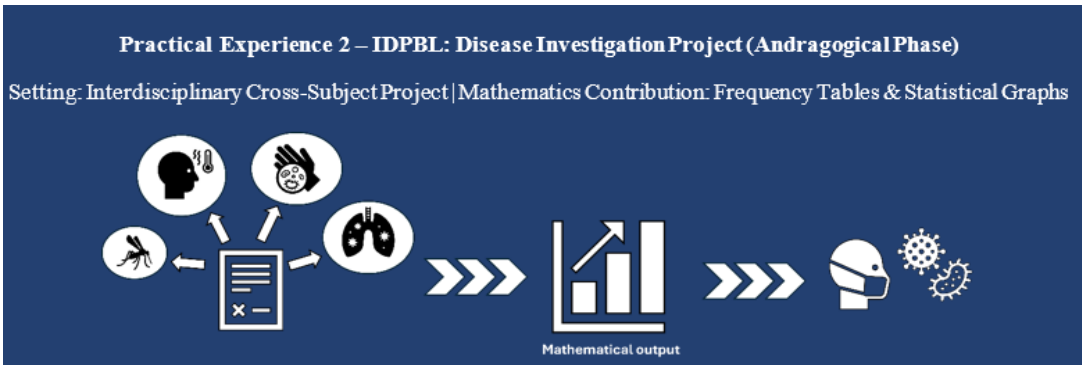

Figure 2: Disease investigation project schematic illustration

The mathematical outputs fed directly into the work of other subjects, such as Science to investigate disease patterns, English and Bahasa Indonesia for analytical and creative writing, Computer Science and Visual Arts, which translated the mathematical visualizations into infographic formats, which would aid in spreading information to various communities, as highlighted from Global Perspectives. This web of interdependence created what Lave and Wenger [22], as cited in Brown et al. [7], called 'authentic peripheral participation', in which students not only gain knowledge by simply absorbing what is given in class, but also by participating in real-world scenarios and applying what is taught. The andragogical approach is highlighted in this experience through the disease outbreak scenario, where students are expected to process the epidemiological data given to visualize both impact and causality, which are then used as a basis for decision-making. This implementation addresses what Boaler [5] identifies as the 'relevance gap' where students lose interest or fail to find its relevance as mathematics lacks contextualization for everyday applications.

2.2.2. Collaboration: Experience 4 — Group Dice Tossing (Probability)

A group-based dice-tossing task is illustrated in Figure 3 below. Groups of four students conducted 120 trials with a standard die, recorded the outcomes on a tally sheet, and then organized the data into frequency tables, bar graphs, pie charts, and pictograms. Using the same data set, students are encouraged to analyze different visualization techniques.

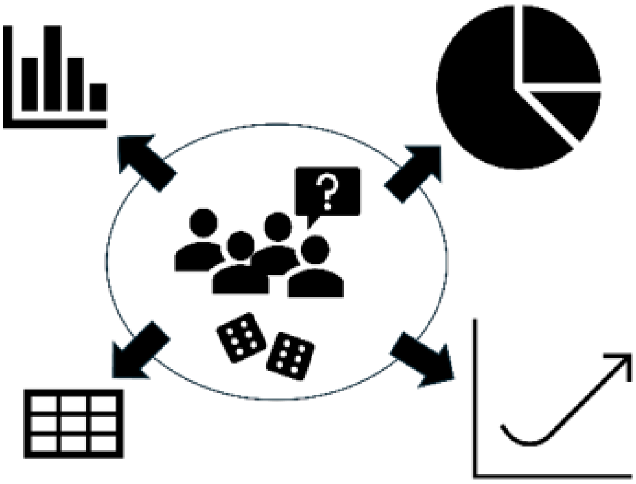

Figure 3: Group dice tossing task illustration

This would ultimately deepen their understanding of statistical representation. Collaboration with other students introduces an important social dimension consistent with andragogical principles. The students' prior experiences with randomness, fairness, and probability intuitions become resources for group discussion and negotiation. The task itself creates physical uncertainty, as it encourages productive group discussions by using the graphs generated. Students receive teacher feedback at the end of class to further refine their work, thereby strengthening their confidence in scientific reasoning and emphasizing the value of the investigative process over the final answer.

2.3. The Heutagogical Leap: Self-Determined Exploration

First theorized by Black and William [3], heutagogy was further developed by Blaschke [4]. It is often challenged by conventional instructional models, especially in education, as students have complete control over learning. The heutagogy approach expects the learners to acquire knowledge themselves while also curating what knowledge to acquire. In secondary mathematics, this approach requires a fundamental trust in the learner's capacity to identify real-world situations worth investigating, mathematically represent their observations through modeling or statistical visualizations, and, finally, answer the questions or gaps that were investigated. Heutagogy is a harmony of interconnected concepts, namely the self-determination theory, the double-loop learning theory, and the capability development theory. Self-determination theory concerns students' motivation to acquire knowledge and gain conceptual depth in the chosen topic, which is strongly correlated with students' capability and the relevance of the knowledge to their lives.

Double-loop learning theory, as proposed by Argyris and Schön [1], as cited in Blaschke [4], is a philosophical approach that encourages further "how" and "why" questions about the applied method they chose to guide their investigation, rather than following what was taught in their pedagogical stage. Whereas the capability development theory distinguishes between competence and capability, the heutagogical approach emphasizes capability as a standard for applying knowledge to unfamiliar contexts. The pedagogical risk of heutagogical approaches is real and must be acknowledged. Without careful design, self-directed inquiry can result in mathematical misconceptions, shallow coverage of formally required depth of knowledge, and inequitable outcomes for students whose home environments provide less support for academic exploration. An effective heutagogical design, therefore, requires a strong foundation laid before students are tasked with exploration. As the students work on their foundation, the teacher steps back and lets them take control of their learning. However, regular consolidation with the teacher is still required, ensuring that mistakes and misconceptions are addressed early and don't accumulate at the end, harming the student's confidence.

2.3.1. Practical Evidence: Experience 3 Simultaneous Linear Equations

The third instructional experience encompasses the transitional phase from andragogical elements (group work, real-world relevance) to heutagogical elements (student-generated content, self-chosen problems). By allowing students to choose their own problem, it pivots the point between andragogy and heutagogy. Teachers should be prepared for the contextual diversity of the problems students select. From fruit market transactions to token earnings in a video game, students are still expected to identify and transform experiential phenomena into two independent relationships through a linear equation, as Freundenthal (1991, as cited in Gravemeijer and Doorman [12] termed horizontal mathematization. At times, the student's chosen scenario creates an imperative dimension in which one variable relates to another in a specific way, a common concept in the STEM field. This envisions students' impact on mathematical thinking in simple, practical solutions (Figure 4).

Practical Experience 3 — Simultaneous Linear Equations (Andragogical-Heutagogical Transition)

Setting: Grade 8 Mathematics | Topic: Simultaneous Linear Equations in Real-World Application

Students worked in groups of four to identify real-world scenarios that could be modeled using simultaneous linear equations. Discovered examples included: purchasing fruit (1.5 kg mango + 2 kg apple vs. 2 kg mango + 1 kg apple; bottled water and chocolate purchasing combinations; tokens in games; etc.

Students then formalized their scenarios into equation pairs and solved them — encouraging the use of mathematical modelling for their individual experience. This combination between scenarios and mathematical formalism is the defining feature of applied mathematical thinking.

Figure 4: Andragogical–heutagogical transition in learning simultaneous linear equations

Vygotsky's [33] concept of the ZPD emphasizes how a personalized experience reduces cognitive load and enhances mathematical understanding, especially when students work in a familiar setting. This pedagogical wisdom aligns with Realistic Mathematics Education, which first allows students to contextualize problems before encountering the formal language of mathematics to model realistic phenomena.

2.3.2. The Pinnacle of Heutagogy: Experience 5 — Direct and Inverse Proportion

The fifth and final instructional experience represents the most fully realized heutagogical learning approach. Students were assigned to independently research and learn the concepts of direct and inverse proportions before the teacher introduced the topic. In this approach, the role of the teacher shifts from a mere instructor to a facilitator, with misinterpretations of the concept corrected after students have conducted independent research. This transformation was described by Boaler [5], who argued that role shifting is necessary in heutagogical learning. As students think about examples where the concepts of inverse and direct proportion apply, this encourages independent learning (Figure 5).

Practical Experience 5 — Direct and Inverse Proportion (Heutagogical Phase)

Setting: Secondary Mathematics | Topic: Direct and Inverse Proportion | Approach: Full Heutagogy

Students independently researched the concepts of direct and inverse proportion prior to any introduction given by the teacher. Self-generated examples included: proportional budgeting for grocery shopping, and recipe scaling (e.g., 2 cups flour for 8 pancakes — how many cups for 26 pancakes?).

Outputs were presented as video presentations accompanied by written reflections. The teacher's after their independent research includes reviewing outputs, correcting misconceptions, and aligning student understanding with formal curriculum definitions. This inversion of the instruction sequence — student discovery preceding teacher intervention — is the defining nature of heutagogy.

Figure 5: Heutagogical learning approach for direct and inverse proportion

Scenarios such as grocery shopping on a given budget or making a dozen pancakes with a recipe for 10 pancakes demonstrate the student's capacity to apply the concept to everyday problems. These scenarios involve ratio and non-integer proportions, which constitute another topic of the mathematics curriculum, highlighting the interconnected nature of learning mathematics. The use of video presentation as the primary output is a significant pedagogical choice consistent with heutagogical principles. Video production requires students to externalize their understanding by explaining, providing logical justifications, and communicating their mathematical reasoning, thereby revealing their true comprehension, which is otherwise hard to assess in written tests. The accompanying written reflections introduce a metacognitive dimension that is deeply reflected in heutagogical approaches. The students themselves determine what they have learned and what they still miss or remain uncertain about. This double-loop reflective practice is what Blaschke [4] defines as the distinguishing features of heutagogical competence. The teacher's role of reviewing outputs, correcting misconceptions, and aligning student understanding with formal curriculum definitions reflects what Kapur [17] terms 'productive failure'. Students are not expected to understand the concept fully, but are encouraged to learn, explore, and identify scenarios independently before correcting to sharpen their understanding. Research shows that this particular approach enhances long-term retention and transfer compared to teacher-based, instruction-first models, especially when the consolidation stage is substantive and well-designed. Thus, this experience demonstrates how heutagogical learning does not compromise mathematical rigor but rather introduces it earlier.

2.4. Technological Integration Across the PAH System

Technology's role in PAH-aligned mathematics education should be given serious consideration. At the pedagogical end of the spectrum, dynamic geometry software such as GeoGebra and Desmos supports teacher-directed visualization, leading to better understanding than just books and presentations. At the andragogical end, Google Workspace and Microsoft Teams are some of the common platforms students can use to coordinate group projects and share their findings. Research tools (including artificial intelligence), video production software, and design platforms such as Canva support the heutagogical design of self-directed queries and subsequent findings, as demonstrated through their communication skills. The integration of the Canva-based infographic in the IDPBL experience is a noteworthy example of how technology can serve as both a learning tool and a communication medium. Students' ability to translate mathematical data (e.g., frequency tables, bar graphs) into visually compelling infographics requires mathematical, communicative, and design competencies that mirror professional data visualization practices.

This convergence of mathematical content with real-world professional skills is precisely the kind of 'twenty-first-century literacy' skills targeted in the professional world, as highlighted by English and Watters [11] and the World Economic Forum's Future of Jobs reports. Research on technology integration in secondary mathematics consistently demonstrates that the pedagogical quality of integration to support mathematical thinking matters far more than the presence or sophistication of technology itself. Technology that positions learners as explorers and creators (consistent with andragogy and heutagogy) yields significantly stronger outcomes than technology that merely digitizes learning through slides and online assignments (a pedagogical failure mode). The PAH framework thus provides a useful evaluative lens for technology integration decisions, prompting educators to ask not only 'what can this technology do?' but 'how can this technology enhance mathematical understanding for each unique individual?'

3. Discussion

3.1. Authority vs Autonomy

One of the most contentious aspects of this review is the teacher-learner power dynamic. The tension is often framed as between "traditional" and "progressive" instruction, yet the 5 instructional experiences highlight the synergistic combination of the two. Effective mathematics instruction requires both the "instructional" and "progressive" modes, in which teachers should fill the gaps in students' understanding. The five experiences illustrate that neither pure pedagogy (Experience 1's structured Pythagorean lesson) nor pure heutagogy (Experience 5's self-directed proportional research) represents a one-size-fits-all learning approach, but rather an optimal design that also requires further tuning of all aspects, such as learner readiness, content complexity, and instructional purpose. This insight is consistent with what Darling-Hammond et al. [8] term 'adaptive expertise' in teaching, the capacity to develop skills to flexibly address students' needs in acquiring knowledge to their potential. The PAH spectrum provides a theoretical map of these requirements, but navigating it effectively requires pedagogical knowledge that is both deep and responsive.

The downfall of the pedagogical end is students who are procedurally fluent but conceptually shallow. In contrast, the heteragogical end is students who are confused, anxious, and have conceptual foundations that can easily be shattered by greater complexity. The art lies in calibrating the progression — moving students along the spectrum at a pace that stretches without overwhelming both teachers and students. Critically, the experiences analyzed here demonstrate that the same teacher can operate at multiple points on the PAH spectrum across different lessons, topics, and student groupings. For example, the

Pythagorean Theorem lesson falls within the domain of precise formulae and mathematical concepts, where the pedagogical model shines. The Direct and Inverse Proportion research paper builds on the extensive prior mathematical expertise that students develop, and introducing it through experiential world scenarios does not undermine these foundations but rather fosters an integrative approach to self-development and learning. The skill in PAH-informed teaching is not adhering to a single point on the spectrum but mastery of the full range and a conscious, informed judgment to move fluidly within it.

3.2. Institutional Barriers and Management

The implementation of PAH-spectrum approaches in secondary mathematics education occurs within institutional contexts that can either actively support or systematically inhibit the shift towards greater learner autonomy. The institutional barriers are analyzed based on the five experiences. First, curriculum frameworks that prioritize content coverage over competency development often lean on pedagogical approaches to target rapid transmission. When external examinations reward procedural recall over applied reasoning, teachers would focus on procedural enforcement rather than on inquiries and student engagement. Second, school timetabling and physical infrastructure would hinder andragogical and heutagogical approaches. Interdisciplinary papers like the IDPBL require coordination across eight subject departments, which demands institutional commitment across departments, shared planning time, and administrative flexibility not commonly available in secondary school timetables. The success of the IDPBL experience suggests that such institutional practices are possible and productive when they are deliberately created and sustainably supported. Third, the teacher's individual development is represented as one of the most critical institutional factors affecting the success of PAH implementation.

Teachers should be aware of which end of the spectrum (pedagogy, andragogy, or heutagogy) is most appropriate for the learning scheme, as this will determine which assessments are valid for each spectrum while consistently maintaining the classroom dynamics. Research shows that isolated professional development workshops that some teachers participate in have little to no effect on their classrooms in an institutional setting. The role of the school leader is highly emphasized, with the leader encouraging collaborative professional communities to develop teachers' teaching skills through workshops or seminars. Fourth, parental and community expectations can create resistance to more autonomous learning, especially when high-stakes examinations are part of the school curriculum. When students and parents view educational success through markers such as grades, homework, and drills, which align with traditional pedagogy, the introduction of research papers and group investigations might generate anxiety and skepticism. Questions such as 'how would this help my child learn?' or 'how would the teacher assess my child's understanding?' arise and create more barriers to more progressive learning. An effective institutional management approach is required to provide open, proactive communication across stakeholders, such as parents, to inform and educate them about the outcomes of each instructional model.

3.3. Student Readiness and Engagement

The five instructional experiences provide compelling evidence for a paper that challenges common assumptions about student engagement in mathematics: student excitement, motivation, and deep engagement. These factors are not only detrimental to the heutagogical and andragogical approaches but also to the pedagogy end of the spectrum. In the 1st instructional experience on the Pythagorean Theorem, the exit ticket was the main highlight, generating enthusiasm and excitement and allowing the teachers themselves to assess whether a full understanding of the concept was achieved during the session. The IDPBL project generated an engaged, interdisciplinary investigation of how real-world problems relevant to their lives can be modeled as simultaneous equations and fostered a mathematical awareness of how one variable relates to another. The highlight of the Direct and Inverse Proportion heutagogical research paper was the prior exploration and understanding of the topic with teacher explanation as an "additional" component to correct misinterpretations. These observations suggest that the quality of engagement is more closely tied to the quality of instructional design across the PAH spectrum rather than to the degree of learner autonomy per se. These findings highlight how educators can frame the PAH spectrum within their mathematics curriculum, where each point on the spectrum is of equal value and elicits different kinds of engagement.

The psychological output of the pedagogical model is a focused, task-directed, and competence-affirming behavior, whereas the andragogical model is more collaborative, contextually rich, and societally meaningful. Finally, the heutagogical model generates a deeply personal, self-referential, and motivational need to acquire knowledge towards an expected communicative result. A mathematics education that offers students only one kind of engagement extinguishes the depth of their mathematical experience. The goal is not to maximize autonomy but to develop the full range of mathematical understanding from disciplined procedural practice to creative problem formulation, which constitutes mathematical fluency in its broadest sense. Student readiness for heutagogical approaches is a legitimate concern that the present experiences illuminate. The progression from Experience 1 (pedagogy) through Experiences 2 and 4 (andragogy) to Experiences 3 and 5 (heutagogy) does not appear to be arbitrary but a stepwise developmental scaffolding of autonomy. Students who have been supported in applying mathematical knowledge to teacher-provided real-world contexts (andragogy) are better positioned to generate their own real-world contexts (heutagogy) than students who move directly from procedural instruction to independent inquiry. This suggests that a proper and guided autonomy through the pedagogical and andragogical phase provides a strong foundation for a successful heutagogy model.

4. Conclusion

This paper has explored the application of the Pedagogy-Andragogy-Heutagogy spectrum in secondary mathematics education through both a theoretical review and practical instructional experience. Drawing on five documented instructional experiences and 38 peer-reviewed sources, the study has established that the PAH spectrum provides a robust and practically actionable framework for understanding and transforming mathematics instruction in secondary schools. The theoretical analysis confirms that each modality in the PAH spectrum — pedagogy, andragogy, and heutagogy — offers a distinctive and irreducible contribution to students' full mathematical development. Pedagogy provides the structured, expert-guided introduction of formal concepts and procedures that constitute the irreplaceable foundation of mathematical knowledge. Andragogy extends this foundation by embedding mathematical tools within relevant, collaborative, and problem-centered contexts that activate self-motivation and deepen conceptual understanding. Heutagogy concludes the developmental progression by repositioning learners as the primary architects of their own mathematical inquiry, as they are developed not only by their competence but also by their ability to apply mathematical thinking independently, from familiar scenarios into unfamiliar contexts. The practical analysis of the five instructional experiences illustrates what PAH-spectrum instruction looks like at the classroom level. Experience 1 demonstrates that excellent pedagogy can generate genuine engagement and learning with guided outcomes.

Experience 2 illustrates the andragogical, interdisciplinary design that connects mathematical competency to broader investigative purposes. Experience 3 reveals the andragogical-heutagogical threshold as students advance from applying mathematics in teacher-provided contexts to generating mathematical problems from their own lived experience. Experience 4 demonstrates how collaborative inquiry develops multi-representational statistical reasoning within an andragogical frame. Experience 5 is the fullest realization of heutagogy, with students researching, discovering, and presenting mathematical knowledge before any formal instruction, and the teacher in the facilitating role of auditor and corrector. The discussion has further addressed the unavoidable tension between authority and autonomy that is associated with PAH-informed teaching, the institutional barriers that must be deliberately addressed to enable spectrum-wide instruction, and the nuanced nature of student engagement across different instructional modalities. Additionally, it was highlighted that learners' autonomy is often associated with engagement quality and that each PAH approach generates and is expected to yield different positive behavioral outcomes for students, further revealing their learning potential. The implications for practice are significant. Secondary mathematics teachers are called to develop what this paper terms 'PAH fluency,' which includes the theoretical understanding of each instructional modality, the practical skills to design effectively within each modality, and the professional judgment to select and sequence approaches in response to their students' developing readiness.

Institutions are called to create the structural conditions, such as flexible timetabling, sustained professional development, aligned assessment design, and stakeholder communication, which enable the PAH-spectrum instruction to flourish. For future research, studies are needed to trace the impact of PAH-informed secondary mathematics instruction on students' mathematical coherence, identity, and performance in higher education, as reflected in this review. Comparative studies between instructional sequences that mirror the five experiences analyzed here and conventional pedagogical sequences would provide valuable evidence of their impact on students' understanding. Research on the role of teacher professional development in supporting PAH transitions would inform institutional strategy. At the same time, qualitative investigations of student experience at the heutagogical end of the spectrum would determine the field and depth of understanding of a self-determined mathematical inquiry. In conclusion, the Pedagogy-Andragogy-Heutagogy spectrum is not a pedagogical ideology but a pedagogical geography in which secondary mathematics teachers can navigate their strategies towards a more diverse learning experience. The evidence reviewed here suggests that teachers who learn to move fluidly and purposefully across this geography perform best at cultivating students who are not merely mathematically knowledgeable but mathematically capable, confident, and curious. These are precisely the graduates of the twenty-first century, who are uniquely positioned to produce critical individuals ready for higher education through PAH-informed secondary mathematics education.

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